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Equilibrium of charges and differential equations solved by polynomials

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Abstract
We study limits of particular importance of the bilinear hypergeometric equation introduced in [8]. As part of this study, we examine connections between the rationality of certain indefinite integrals and the equilibrium of Coulomb charges in the complex plane (or point vortices in two-dimensional hydrodynamics). Relationships with integrable models which are generalizations of the Calogero–Moser systems are also discussed.

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1. Introduction

The bilinear hypergeometric equation is a generalization of the Gauss hypergeometric equation [6] and the bilinear differential equation for the Adler–Moser polynomials [1]. It has appeared in its generic form in [8] in connection with integrable dynamics of certain systems.

In the present work, we consider its limits of particular importance related to the equilibrium of free Coulomb charges (or point vortices) in the complex plane. We also introduce integrable dynamical systems for which these equilibrium configurations are fixed points.

Instead of, considering directly different limits of the generic equation [6], we start with a more instructive and mathematically interesting approach similar to [3].

In their 1929 paper, Burchnall and Chaundy [3] examined the following (apparently elementary) question: what condition must be satisfied by two polynomials $p(z)$ and $q(z)$ in one variable z in order that the indefinite integrals

$$\int \left(\frac{p(z)}{q(z)}\right)^2 \mathrm{d}z, \qquad \int \left(\frac{q(z)}{p(z)}\right)^2 \mathrm{d}z \tag{1}$$

may be rational, i.e. expressible without logarithms. Under the assumption that p and q have no common or repeated factors, the problem is reduced to finding polynomial solutions of the following bilinear differential equation:

$$p''q - 2p'q' + pq'' = 0. \tag{2}$$

It was shown that solutions to (2) are Adler–Moser polynomials (polynomial τ-functions of the KdV equation) [1]:

$$p = \theta_i \qquad q = \theta_{i+1} \qquad \theta_0(z) = 1 \qquad \theta_1(z) = z \qquad \deg \theta_i = \frac{i(i+1)}{2}. \tag{3}$$

Equation (2) can be viewed as a second-order (two-term) differential-recurrence relation for the the Adler–Moser polynomials θ_i, θ_{i+1}. It can be rewritten as the first-order (three-term) relation [1, 3]

$$\theta'_{n+1}\theta_{n-1} - \theta_{n+1}\theta'_{n-1} = (2n+1)\theta_n^2. \tag{4}$$

Considering (4) as a first-order linear differential equation for θ_{n+1}, we obtain recursively all solutions introducing an integration constant at each step. It is important to stress that the solutions of this differential equation are polynomials due to the rationality of integrals (1).

Thus, the nth polynomial depends on n free parameters

$$\theta_n = \theta_n(z; t_1, t_2, \ldots, t_n). \tag{5}$$

Several first examples of the Adler–Moser polynomials are as follows:

$$\begin{aligned} &\theta_0 = 1 \\ &\theta_1 = z \\ &\theta_2 = z^3 + t_2 \\ &\theta_3 = z^6 + 5t_2z^3 + t_3z - 5t_2^2 \\ &\vdots \end{aligned} \tag{6}$$

In (6), we set $t_1 = 0$, since t_1 can be absorbed by the translation $z \to z + t_1$.

It was observed in [2] that roots of the consecutive polynomials θ_i, θ_{i+1} are the equilibrium coordinates of $i(i+1)/2$ positive and $(i+1)(i+2)/2$ negative Coulomb charges (with values ± 1 respectively) on the plane interacting through a two-dimensional (logarithmic Coulomb) potential. Namely, the function

$$E = \sum_{i<j=1}^{n(n+1)/2} \ln|x_i - x_j| + \sum_{i<j=1}^{(n+1)(n+2)/2} \ln|y_i - y_j| - \sum_{i=1}^{n(n+1)/2} \sum_{j=1}^{(n+1)(n+2)/2} \ln|x_i - y_j|$$

has a critical point when x_i and y_i are roots of θ_n and θ_{n+1} respectively.

It must be mentioned that, although relations between zeros of different kinds of (classical orthogonal) polynomials and the equilibrium of identical charges (in different external fields) on the real line have been known since the works by Sego [11], similar questions of equilibrium of nonidentical charges have not been addressed.

The constructive way to find the Adler–Moser polynomials is to use their determinant representation constructed by means of the Darboux–Crum transformations of the operator $\mathrm{d}^2/\mathrm{d}z^2$.

In more detail, equation (2) can be rewritten in the two equivalent Schrödinger forms (recall that $p = \theta_i, q = \theta_{i+1}$)

$$H_i\left[\frac{\theta_{i+1}}{\theta_i}\right] = 0 \qquad \text{or} \qquad H_{i+1}\left[\frac{\theta_i}{\theta_{i+1}}\right] = 0 \qquad H_i = \frac{\mathrm{d}^2}{\mathrm{d}z^2} + 2(\ln(\theta_i))''.$$

The two representations of (2) above are connected by permutation of the first-order factors in the second-order differential operators H_i.

Using this observation we can extend, the factorization chain by induction, obtaining the Adler–Moser polynomials at each step. The following figure demonstrates steps of this procedure:

$$\begin{array}{lllll} \downarrow & & & & \\ H_i = L_i M_i & = M_{i+1} L_{i+1} & & & \\ & \downarrow & & & \\ & H_{i+1} = L_{i+1} M_{i+1} & = M_{i+2} L_{i+2} & & \\ & & \downarrow & & \\ & & H_{i+2} = L_{i+2} M_{i+2} = & \cdots & \end{array} \tag{7}$$

where

$$L_i = \frac{\theta_i(z)}{\theta_{i-1}(z)} \frac{\mathrm{d}}{\mathrm{d}z} \frac{\theta_{i-1}(z)}{\theta_i(z)} \qquad M_i = \frac{\theta_{i-1}(z)}{\theta_i(z)} \frac{\mathrm{d}}{\mathrm{d}z} \frac{\theta_i(z)}{\theta_{i-1}(z)}$$

and $L_0 = M_0 = \mathrm{d}/\mathrm{d}z$.

In (7), arrows denote transitions (Darboux transformations) from the second-order differential operator H_i to H_{i+1} by the permutation of factors M_{i+1}, L_{i+1} while, between these transitions, we vary the factors $L_i \to M_{i+1}$, $M_i \to L_{i+1}$, keeping H_i unchanged. This variation of factors is possible due to the freedom in choosing linearly independent solutions of the second-order operator H_i, which amounts to acquiring a free parameter at each level of factorization (see (5)).

From (7), we derive the intertwining differential operators D_i, U_i

$$U_i \frac{\mathrm{d}^2}{\mathrm{d}z^2} = H_i U_i \qquad \frac{\mathrm{d}^2}{\mathrm{d}z^2} D_i = D_i H_i \qquad D_i = M_i \cdots M_1 \qquad U_i = L_1 \cdots L_i \tag{8}$$

connecting $H_0 = \mathrm{d}^2/\mathrm{d}z^2$ with H_i.

This, together with the obvious property $D_i U_i = \frac{\mathrm{d}^{2i}}{\mathrm{d}z^{2i}}$, leads to the Wronskian representation for the Adler–Moser polynomials. In more detail [3, 1]

$$\theta_n = \text{const}\, W[\psi_1, \ldots, \psi_n] \qquad \psi_n'' = \psi_{n-1} \qquad \psi_1 = z \tag{9}$$

where the nth polynomial is (up to a constant factor) the Wronskian $W = \det \|\mathrm{d}^i \psi_j / \mathrm{d}z^i\|_{0 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant n}$.

The intertwining operators are also expressible in terms of Wronskians. For instance

$$U_i[\psi] = W[\psi_1, \ldots, \psi_i, \psi] / W[\psi_1, \ldots, \psi_i]. \tag{10}$$

2. Indefinite integrals related to equilibrium of charges

By analogy with the previous section, we pose the following question: when is an equilibrium condition for charges of different sign equivalent to rationality of some indefinite integrals?

Before addressing this question we write the equilibrium equation for two types of charges in terms of polynomials.

Lemma 1. *Let*

$$E = \sum_{1 \leqslant i < j \leqslant n+m} Q_i Q_j \ln |z_i - z_j| \tag{11}$$

be a real function of $n + m$ complex variables $z_i \in \mathbb{C}$, $i = 1, \ldots, n + m$ and

$$Q_i \in \mathbb{R} \qquad Q_i = \begin{cases} 1 & i = 1, \ldots, n \\ -\Lambda & i = n + 1, \ldots, m + n \end{cases}$$

Then E has a critical point at

$$z_i = \begin{cases} x_i & i = 1, \ldots, n \\ y_{i-n} & i = n+1, \ldots, m+n \end{cases}$$

iff

$$\{p, q\}_\Lambda := \frac{\mathrm{d}^2 p(z)}{\mathrm{d}z^2} q(z) - 2\Lambda \frac{\mathrm{d}p(z)}{\mathrm{d}z}\frac{\mathrm{d}q(z)}{\mathrm{d}z} + \Lambda^2 p(z) \frac{\mathrm{d}^2 q(z)}{\mathrm{d}z^2} = 0 \tag{12}$$

where p, q *are the following polynomials:*

$$p(z) = \prod_{i=1}^{n} (z - x_i) \qquad q(z) = \prod_{i=1}^{m} (z - y_i).$$

Proof. The proof can be shown by a calculation. We use the partial-fraction decomposition of $\frac{\{p,q\}_\Lambda}{pq}$ around singular points x_i and y_i. □

Now we are in position to find an equivalent condition for p and q in terms of rational integrals.

Indefinite integrals of the algebraic function do not contain logarithmic terms if the function does not have singularities of the type z^{-1}. The following lemma shows when certain functions, related to solutions of (12), are free of simple poles and their integrals are rational.

Lemma 2. *Let* $p(z) = \prod_{i=1}^{n}(z - x_i), q(z) = \prod_{i=1}^{m}(z - y_i)$ *not have multiple or common roots. Then residues of simple poles of* $q^{2\Lambda}/p^2$ *and* $p^{2/\Lambda}/q^2$ *vanish:*

$$Res_{z=x_i} \frac{q(z)^{2\Lambda}}{p(z)^2} = 0 \qquad i = 1, \ldots, n \qquad Res_{z=y_i} \frac{p(z)^{2/\Lambda}}{q(z)^2} = 0 \qquad i = 1, \ldots, m$$

iff p and q satisfy (12).

Both indefinite integrals

$$\int \frac{q(z)^{2\Lambda}}{p(z)^2}\, \mathrm{d}z, \qquad \int \frac{p(z)^{2/\Lambda}}{q(z)^2}\, \mathrm{d}z \tag{13}$$

are rational iff p, q *satisfy (12) with* $\Lambda = 1/2, 1, 2$.

Proof. Let us factorize $p(z)$ as $p(z) = (z - x_i)P(z)$. Then the condition $Res_{z=x_i} \frac{q(z)^{2\Lambda}}{p(z)^2} = 0$ implies that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{q(z)^{2\Lambda}}{P(z)^2}\right)_{z=x_i} = 0.$$

Since $P(x_i) = p'(x_i)$, $P'(x_i) = p''(x_i)/2$, we obtain the following equation:

$$p''(x_i)q(x_i) - 2\Lambda p'(x_i)q'(x_i) = 0.$$

Thus, since x_i are roots of $p(z)$,

$$p''(z)q(z) - 2\Lambda p'(z)q'(z) = a(z)p(z) \tag{14}$$

where $a(z)$ is a polynomial.

Similarly, from the condition $Res_{z=y_i} \frac{p(z)^{2/\Lambda}}{q(z)^2} = 0$, we obtain

$$\Lambda^2 p(z)q''(z) - 2\Lambda p'(z)q'(z) = b(z)q(z). \tag{15}$$

Adding $\Lambda^2 p(z)q''(z)$ to (14) and $p''q(z)$ to (15), we get

$$p''(z)q(z) - 2\Lambda p'(z)q'(z) + \Lambda^2 p(z)q''(z) = A(z)p(z) = B(z)q(z)$$

where $A(z) = a(z) + \Lambda^2 q''(z)$, $B(z) = b(z) + p''(z)$.

Since $p(z)$ and $q(z)$ do not have common roots, it follows from

$$A(z)p(z) = B(z)q(z)$$

that $A(z) = N(z)q(z)$, $B(z) = N(z)p(z)$, where $N(z)$ is a polynomial. Thus, we arrive at

$$p''(z)q(z) - 2\Lambda p'(z)q'(z) + \Lambda^2 p(z)q''(z) = N(z)p(z)q(z). \tag{16}$$

The degree of the polynomial on the left-hand side of (16) is at most $\deg(q)+\deg(p)-2$ while the degree on the right-hand side is at least $\deg(q) + \deg(p)$, unless $N(z) = 0$. Therefore, $N(z) = 0$ and we obtain (12).

The choice $\Lambda = 1/2$, $\Lambda = 1$, $\Lambda = 2$ is essential for the rationality of (13) because both exponents 2Λ and $2/\Lambda$ are integers. This completes the proof. □

Since the cases $\Lambda = 1/2$ and $\Lambda = 2$ are equivalent (one is connected with the other by permutation of p with q), we have two second-order bilinear differential relations corresponding to two types of rational integrals

- Adler–Moser polynomials, $\Lambda = 1$ (see [3]),

$$\begin{aligned} &\{p, q\}_1 = \frac{\mathrm{d}^2 p(z)}{\mathrm{d}z^2} q(z) - 2\frac{\mathrm{d}p(z)}{\mathrm{d}z}\frac{\mathrm{d}q(z)}{\mathrm{d}z} + p(z)\frac{\mathrm{d}^2 q(z)}{\mathrm{d}z^2} = 0 \\ &\int \frac{q(z)^2}{p(z)^2}\,\mathrm{d}z, \qquad \int \frac{p(z)^2}{q(z)^2}\,\mathrm{d}z. \end{aligned} \tag{17}$$

- $\Lambda = 2$,

$$\begin{aligned} &\{p, q\}_2 = \frac{\mathrm{d}^2 p(z)}{\mathrm{d}z^2} q(z) - 4\frac{\mathrm{d}p(z)}{\mathrm{d}z}\frac{\mathrm{d}q(z)}{\mathrm{d}z} + 4p(z)\frac{\mathrm{d}^2 q(z)}{\mathrm{d}z^2} = 0 \\ &\int \frac{q(z)^4}{p(z)^2}\,\mathrm{d}z, \qquad \int \frac{p(z)}{q(z)^2}\,\mathrm{d}z. \end{aligned} \tag{18}$$

Equation $\{p, q\}_2 = 0$ has been mentioned in [2] in connection with the problem of equilibrium of the point vortices in two-dimensional hydrodynamics.

The following sections are devoted to the investigation of the second case and its generalizations.

3. Λ = 2

We begin by summing up what we have obtained so far.

Let $p(z)$ and $q(z)$ be polynomials of the complex variable z which do not have common or multiple roots.

We study the following three problems.

- When are the indefinite integrals

$$\int \frac{p}{q^2}\,\mathrm{d}z, \qquad \int \frac{q^4}{p^2}\,\mathrm{d}z \tag{19}$$

  rational?
- When can the system of n and m Coulomb charges of values 1 and -2 respectively be in equilibrium in the complex plane and what are the coordinates of the charges?
  In other words, when does the energy function

$$E = \sum_{i<j} Q_i Q_j \log|z_i - z_j| \qquad Q_i = \begin{cases} 1 & i = 1, \ldots, n \\ -2 & i = n+1, \ldots, n+m \end{cases} \tag{20}$$

  have a critical point and at which z_i, $i = 1, \ldots, n+m$?

- When does the equation

$$\{p, q\}_2 = 0 \tag{21}$$

have polynomial solutions?

The answer to the above three questions is given by the following proposition.

Proposition 1. *Let $i \in \mathbb{Z}$, and (modulo translation of z) $p_{-1} = z, \quad p_0 = q_0 = 1, \quad q_1 = z$. Then*

- *all solutions of the recurrence relations*

$$\begin{aligned} q'_{i+1}q_i - q_{i+1}q'_i &= (3i+1)p_i \\ p'_i p_{i-1} - p_i p'_{i-1} &= (6i-1)q_i^4 \end{aligned} \tag{22}$$

are polynomials of degrees

$$\deg(p_i) = i(3i+2) \qquad \deg(q_i) = i(3i-1)/2 \qquad i \in \mathbb{Z}.$$

They provide all solutions of (21)

$$\{p_i, q_i\}_2 = \{p_i, q_{i+1}\}_2 = 0 \qquad i \in \mathbb{Z}.$$

- *Integrals (19) are rational iff p and q are connected by the bilinear equation (21), i.e. iff*

$$p = p_i \qquad q = q_i \qquad \text{or} \qquad p = p_i \qquad q = q_{i+1}.$$

- *The energy function (20) has a critical point provided z_i are zeros of the above pairs of polynomials: zeros of p and q being positions of charges 1 and -2 respectively.*

Proof.

- The equivalence between the bilinear equation (21) and rationality of integrals (19) is a corollary of lemma 2.
- The equivalence between (21) and the existence of critical points of the energy function (20) is a corollary of lemma 1.
- Since $p = z^n + \cdots$, $q = z^m + \cdots$, from the highest symbol of (21) we get the Diophantine equation connecting n with m

$$(n-2m)^2 - n + 4m = 0.$$

Its solutions are

$$n = n_i \qquad m = m_i \qquad \text{or} \qquad n = n_i \qquad m = m_{i+1} \qquad i \in \mathbb{Z}$$

where

$$n_i = i(3i+2) \qquad m_i = \frac{i(3i-1)}{2}.$$

Let $\{p_i, q_i\}_2 = 0$. Considering this equation as a second-order differential equation with a solution q_i, by elementary methods we find that its second linearly independent solution is given by

$$q_{i+1} = (3i+1)q_i \int \frac{p_i}{q_i^2}\,\mathrm{d}z \qquad \{p_i, q_{i+1}\}_2 = 0 \tag{23}$$

and is a polynomial by the rationality of (19). The degrees of polynomials are connected by

$$\deg(q_{i+1}) + \deg(q_i) = \deg(p_i) + 1.$$

It is seen that this relation is satisfied if $\deg(p_i) = n_i$, $\deg(q_i) = m_i$.

A similar procedure holds if we fix q_i and consider p_{i-1} and p_i as linearly independent solutions of (21):

$$p_i = (6i-1)p_{i-1}\int \frac{q_i^4}{p_{i-1}^2}\,\mathrm{d}z. \tag{24}$$

Having freedom in choosing linearly independent solutions of (21), we write analogues of (23) and (24) for decreasing indices

$$q_i = -(3i+1)q_{i+1}\int \frac{p_i}{q_{i+1}^2}\,\mathrm{d}z \qquad p_{i-1} = -(6i-1)p_i\int \frac{q_i^4}{p_i^2}\,\mathrm{d}z \tag{25}$$

and we can generate p_i, q_i by induction in either direction starting at some i.
Since n_i and m_i are strictly increasing for $i \geqslant 0$ (strictly decreasing for $i \leqslant 0$) and $m_0 = n_0 = 0$, the induction terminates for these two branches at $p_0 = q_0 = 1$.
Rewriting (23), (24) or (25) in differential form, we get the first-order recurrence relations (22), which completes the proof. □

Here we write several examples of polynomials satisfying the above conditions for the branch $i \geqslant 0$

$$\begin{array}{ll} q_0 = 1 & p_0 = 1 \\ q_1 = z & p_1 = z^5 + t_1 \\ q_2 = z^5 + \tau_2 z - 4t_1 & p_2 = z^{16} + \cdots \\ \vdots & \vdots \end{array}$$

and for the branch $i \leqslant 0$

$$\begin{array}{ll} p_0 = 1 & q_0 = 1 \\ p_{-1} = z & q_{-1} = z^2 + \tau_{-1} \\ p_{-2} = z^8 + \frac{28}{5}\tau_{-1}z^6 + 14\tau_{-1}^2 z^4 & q_{-2} = z^7 + 7\tau_{-1}z^5 + 35\tau_{-1}^2 z^3 \\ +28\tau_{-1}^3 z^2 + t_{-2}z - 7\tau_{-1}^4 & +\tau_{-2}z^2 - 35\tau_{-1}^3 z + \tau_{-1}\tau_{-2} - \frac{5}{2}t_{-2} \\ \vdots & \vdots \end{array}$$

where t_i and τ_i represent arbitrary parameters.

4. Intertwining and factorization

One can try to find polynomials for $\Lambda = 2$ explicitly by analogy with the Adler–Moser case through a factorization similar to (7)–(9).

It is easy to observe that equation (21) can be written in two different Schrödinger forms

$$S\left[\frac{p}{q^2}\right] = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 6(\ln q)''\right)\left[\frac{p}{q^2}\right] = 0 \tag{26a}$$

$$\tilde{S}\left[\frac{q}{\sqrt{p}}\right] = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{3}{4}(\ln p)''\right)\left[\frac{q}{\sqrt{p}}\right] = 0 \tag{26b}$$

connected by the following factorization:

$$\frac{p}{q^2}S = LM \qquad \frac{p}{q^2}\tilde{S} = ML \qquad L = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\sqrt{p}}{q} \qquad M = \frac{p^{3/2}}{q^3}\frac{\mathrm{d}}{\mathrm{d}z}\frac{q^2}{p}.$$

Since $p = p_{i-1}$ and $p = p_i$ are solutions of (21) with $q = q_i$, we can put any of them in (26a) with $q = q_i$. A similar fact holds for (26b), with $q = q_i$ or $q = q_{i+1}$ and $p = p_i$.

This ambiguity in the choice of solutions results in the possibility of different factorizations for the same second-order operator. We recall that a similar possibility led to the explicit representation of the Adler–Moser polynomials for $\Lambda = 1$.

By analogy with (7), the factorization scheme has the following form:

$$
\begin{array}{lll}
\downarrow & & \\
\dfrac{p_{i-1}}{q_i^2}S_i = \tilde{L}_i \tilde{M}_i \quad \rightarrow & \dfrac{p_i}{q_i^2}S_i = L_i M_i & \\
& \downarrow & \\
& \dfrac{p_i}{q_i^2}\tilde{S}_i = M_i L_i \quad \rightarrow & \dfrac{p_i}{q_{i+1}^2}\tilde{S}_i = \tilde{M}_{i+1}\tilde{L}_{i+1} \\
& & \downarrow \\
& & \dfrac{p_i}{q_{i+1}^2} S_{i+1} = \tilde{L}_{i+1}\tilde{M}_{i+1} \quad \rightarrow
\end{array} \tag{27}
$$

where

$$\tilde{L}_i = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\sqrt{p_{i-1}}}{q_i} \qquad \tilde{M}_i = \frac{p_{i-1}^{3/2}}{q_i^3}\frac{\mathrm{d}}{\mathrm{d}z}\frac{q_i^2}{p_{i-1}} \qquad L_i = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\sqrt{p_i}}{q_i} \qquad M_i = \frac{p_i^{3/2}}{q_i^3}\frac{\mathrm{d}}{\mathrm{d}z}\frac{q_i^2}{p_i}.$$

The horizontal arrows in (27) correspond to factorizations connected with different choices of p at fixed q and vice versa, while the vertical arrows correspond to permutation of factors. Comparing (27) with (7), we see horizontal arrows instead of equalities. Different from (7), the variation of factors at each level of 'Darboux' transformation is now accompanied by a change of prefactors in front of the Schrödinger operators S_i. Due to this change we do not have simple intertwining relations between $S_0 = \mathrm{d}^2/\mathrm{d}z^2$ and S_i: now, there are two different U^L, U^R (left, right) intertwining operators connecting S_0 with S_i instead of one,

$$U_i^L S_0 = S_i U_i^R.$$

This makes the line of approach used for the Adler–Moser polynomials unpromising for $\Lambda = 2$.

5. Baker–Akhieser functions, charges in homogeneous field

Consider a modified problem: when are the following indefinite integrals

$$\exp(-kz)\int \frac{p(k,z)}{q(k,z)^2}\exp(kz)\,\mathrm{d}z \qquad \exp(2kz)\int \frac{q(k,z)^4}{p(k,z)^2}\exp(-2kz)\,\mathrm{d}z \tag{28}$$

rational? In (28), k is any complex number.

We call the function

$$\Psi(k,z) = \frac{p(k,z)}{q(k,z)^2}\exp(kz) \tag{29}$$

the rational Baker-Akhieser function for $\Lambda = 2$ by analogy with the $\Lambda = 1$ case (see below).

Similar to section 3 we can state the following.

Proposition 2. *Suppose that polynomials $p(k,z)$, $q(k,z)$ in z do not have repeated or common roots. The following three statements are equivalent:*

- *the indefinite integrals*

$$\exp(-kz)\int \Psi(k,z)\,\mathrm{d}z, \qquad \exp(2kz)\int \Psi(k,z)^{-2}\,\mathrm{d}z$$

are rational in z functions, expressible without logarithms.

- *The energy function*

$$E = k \sum_i Q_i z_i + \sum_{i<j} Q_i Q_j \ln |z_i - z_j|$$
$$Q_i = \begin{cases} 1 & i = 1, \dots, n \\ -2 & i = n+1, \dots, m+n \end{cases} \qquad n = \deg(p) \quad m = \deg(q) \tag{30}$$

has a critical point at

$$z_i = \begin{cases} x_i(k) & i = 1, \dots, n \\ y_{i-n}(k) & i = n+1, \dots, m+n \end{cases}$$

where x_i, y_i are roots of

$$p = \prod_{i=1}^{n} (z - x_i(k)) \qquad q = \prod_{i=1}^{m} (z - y_i(k)).$$

- *p, q satisfy the following equation:*

$$0 = \{p, q\}_2^{(k)} := \{p, q\}_2 + 2k(p'q - 2q'p). \tag{31}$$

Proof. The proof follows arguments similar to lemmas 1 and 2. □

One can easily see that equation (30) describes equilibrium of two types of charges with values 1, −2 in the homogeneous electric field.

Another observation consists in the fact that

$$p(k, z) = p(\zeta) \qquad q(k, z) = q(\zeta) \qquad \zeta = kz. \tag{32}$$

Indeed, one can easily check this by scaling z_i, $i = 1, \dots, n+m$ in (30), or by changing the integration variable $z \to \zeta = kz$ in (28).

Thus, we can set $k = 1$ whenever $k \neq 0$.

Equating highest symbols in (31), we obtain the following simple lemma.

Lemma 3. *Let p, q satisfy conditions of proposition 2, then their degrees are related by*

$$\deg(p) = n = 2m = 2\deg(q).$$

In other words, the total charge has to be zero in order that the system is at rest in the homogeneous field of magnitude k.

Although the case $\Lambda = 2$ has some similarities with $\Lambda = 1$, the same approach to explicit representation of p_i, q_i, as shown in section 4, no longer applies.

Here we write several first examples of $p_i(k, z)$, $q_i(k, z)$:

$$\begin{aligned}
&q_0(k, z) = 1 \qquad p_0(k, z) = 1 \\
&q_1(k, z) = \zeta \qquad p_1(k, z) = \zeta^2 - 3\zeta + 3 \\
&q_2(k, z) = \zeta^3 + t_2\zeta^2 + \frac{t_2^2 + 6}{3}\zeta \\
&p_2(k, z) = \zeta^6 + (-9 + 2t_2)\,\zeta^5 + \left(40 - 15t_2 + \tfrac{5}{3}t_2^2\right)\zeta^4 \\
&\qquad + \left(-96 + 52t_2 - 10t_2^2 + \tfrac{2}{3}t_2^3\right)\zeta^3 + \left(112 - 90t_2 + \tfrac{76}{3}t_2^2 - 3t_2^3 + \tfrac{1}{9}t_2^4\right)\zeta^2 \\
&\qquad + \left(-48 + 66t_2 - 28t_2^2 + 5t_2^3 - \tfrac{1}{3}t_2^4\right)\zeta - 18t_2 + 10t_2^2 - 3t_2^3 + \tfrac{1}{3}t_2^4 + 48
\end{aligned} \tag{33}$$

where t_2 is an arbitrary constant.

It is easy to show that solution (33) is generic for $m = 3$, in the sense that it is not contained in a wider class. In other words, coefficients of (properly normalized) q_2 cannot depend on more than one free parameter.

Indeed, we can always take q_2 to be monic and without a constant term (by fixing one root at $z = 0$). If q_2 depended on more than one parameter, then it would be $q_2 = z^3 + t_2 z^2 + Tz$, where t_2 and T are arbitrary constants. In such a case (31) becomes a linear equation for the coefficients of p_2. It is verified by elementary linear algebra that this system is incompatible. Thus q_2 can depend only on one free parameter (say t_2).

It is difficult to obtain examples of polynomials of higher degrees, even with the help of a computer. It is also a nontrivial problem to define degrees of $q(k, z)$. If one assumes that by analogy with the $\Lambda = 1$ case, the order of $q_i(k, z)$ is a quadratic function of i (which certainly holds for $k = 0$, see proposition 1), then it follows from the above examples that $\deg(q_i(k, z)) = i(i + 1)/2$.

6. Scaling, reparametrization and bispectrality

A statement similar to proposition 2 holds for the case $\Lambda = 1$ with the Baker–Akhieser function $\psi = (p/q)\exp(kz)$ instead of (29) and symmetric rationality conditions for the indefinite integrals

$$\exp(-kz)\int \psi\,\mathrm{d}z \qquad \exp(kz)\int \psi^{-1}\,\mathrm{d}z.$$

The electrostatic analogy describes charges 1, -1 in a homogeneous field and equation (31) is replaced by

$$0 = \{p, q\}_1 + 2k(p'q - q'p).$$

This equation can be rewritten in the Schrödinger form

$$L\psi(k, z) = \left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V\right)\psi(k, z) = k^2\psi(k, z) \qquad V = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log q. \tag{34}$$

It follows from (8) and (10) (see [1] for more details) that the solution of (34) is obtained by the action of the intertwining operator (10) on the eigenfunction $\exp(kz)$ of the free Schrödinger operator $\mathrm{d}^2/\mathrm{d}z^2$. This solution is the Baker–Akhieser function:

$$\psi = W[\psi_1, \dots, \psi_n, \exp(kz)]/W[\psi_1, \dots, \psi_n].$$

Therefore, according to (9)

$$p(k, z) = \exp(-kz)W[\psi_1, \dots, \psi_n, \exp(kz)] \qquad q(k, z) = W[\psi_1, \dots, \psi_n] = \mathrm{const}\,\theta_n(z).$$

On the other hand, by arguments leading to (32), we see that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + V\right)\psi(\zeta) = \psi(\zeta) \qquad V = 2\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\log q \tag{35}$$

with

$$\psi(\zeta) = \frac{p(\zeta)}{q(\zeta)}\exp(\zeta) \qquad p(\zeta) = \exp(-\zeta)W[\psi_1(\zeta), \dots, \psi_n(\zeta), \exp(\zeta)]$$
$$q(\zeta) = \mathrm{const}\,\theta_n(\zeta).$$

We observe that potential $V = V(z)$ in (34) is independent of k, while potential $V = V(\zeta)$, in (35), depends on k. This is due to the fact that rescaling of variable $z \to \zeta = kz$ can be absorbed by appropriate changes of the parameters (see e.g. (6))

$$\theta_n(z; t_1, t_2, \dots, t_n) = k^{-n(n+1)/2}\theta_n(kz; kt_1, k^3t_2, \dots, k^{2n-1}t_n)$$

since the Adler–Moser polynomials can be seen as homogeneous polynomials of variables z and t_i, $i > 0$ with weights 1 and $2i - 1$ correspondingly.

Thus the problem of equilibrium of charges 1 and -1 can be reduced to the spectral problem for the k-independent operator L (34).

Similarly, one can ask if $\psi(k, z)$ is simultaneously a solution of a spectral problem for a z-independent differential in k operator A,

$$A\psi(k,z) = \Theta(z)\psi(k,z) \qquad \partial A/\partial z = 0$$

where $\Theta(z)$ is a function of z.

It turns out [5] that such operators exist and belong to the so-called odd bispectral family.

In general, the problem of finding functions satisfying simultaneously a differential equation in z with k-dependent eigenvalues and a differential equation in k with z-dependent eigenvalues is called the bispectral problem [5].

Now returning to the main subject of this work, one might expect that the $\Lambda = 2$ case can belong to some other bispectral (e.g. even [5]) family of the differential operators. Unfortunately, the $\Lambda = 2$ case is not, in general, related to the bispectral problem for a second-order differential operator. Indeed, equation (31) can be rewritten in the following two forms:

$$\begin{aligned}
&\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + u(\zeta)\right)\Psi(\zeta) = \Psi(\zeta) \qquad u(\zeta) = 6\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\log q(\zeta) \qquad \Psi(\zeta) = \frac{p(\zeta)}{q(\zeta)^2}\exp(\zeta) \\
&\left(\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + v(\zeta)\right)\Psi(\zeta)^{-1/2} = \frac{1}{4}\Psi(\zeta)^{-1/2} \qquad v(\zeta) = \frac{3}{4}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\log p(\zeta)
\end{aligned} \tag{36}$$

We use (33) as a counterexample to the bispectrality. It is seen that there is no change of parameter $t_2 \to f(t_1, t_2)$ combined with the translation $z \to z + t_1$ in (33), such that p_2 or q_2 becomes homogeneous (with some weights) in z and t_2. Thus, different from the $\Lambda = 1$ case, rescaling of variable z cannot be absorbed by a change of parameter t_2. As a consequence, potentials u, v in (36) cannot be z or k independent. Therefore, it is impossible, in general, to set a bispectral problem with the second-order differential operator in z (or k) when $\Lambda = 2$.

7. Integrable dynamics of charges

The charge configurations studied above can be viewed as fixed points of some dynamical systems [8].

Lemma 4. *Let two polynomials* $p(z,t) = \prod_{i=1}^{n}(z - z_i(t))$ *and* $q(z,t) = \prod_{i=n+1}^{n+m}(z - z_i(t))$ *satisfy the following bilinear equation:*

$$q\frac{\mathrm{d}p}{\mathrm{d}t} - \Lambda p\frac{\mathrm{d}q}{\mathrm{d}t} = \{p, q\}_\Lambda. \tag{37}$$

Then the roots $z_i(t)$, $i = 1, \ldots, n+m$, *satisfy the system of ordinary differential equations*

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = \sum_{j\neq i=1}^{n+m} \frac{Q_j}{z_i - z_j} = \frac{1}{Q_i}\frac{\partial E}{\partial z_i} \tag{38}$$

where

$$Q_i = \begin{cases} 1 & i = 1, \ldots, n \\ -\Lambda & i = n+1, \ldots, m+n. \end{cases}$$

It follows from the above lemma that the critical points of energy (11) are fixed points of (38). Although, as seen from the previous considerations, the existence of such critical points depends on values m, n and Λ, equation (37) has t-dependent polynomial solutions for any integer $m, n \geqslant 0$ and real Λ.

Although (38) (or equivalently (37)) is not a Hamiltonian system, it can be embedded in the Hamiltonian flow.

Proposition 3. *The dynamical system (38) is a trajectory of a system with the Hamiltonian*

$$H = \frac{1}{2}\sum_{i=1}^{n+m} Q_i \left(\frac{\mathrm{d}z_i}{\mathrm{d}t}\right)^2 + \sum_{i<j=1}^{n+m} \frac{Q_i Q_j (Q_i + Q_j)}{(z_i - z_j)^2}. \tag{39}$$

In other words, the Hamiltonian equations of motion

$$\frac{\mathrm{d}^2 z_i}{\mathrm{d}t^2} = \sum_{j\neq i=1}^{n+m} \frac{Q_j (Q_i + Q_j)}{(z_i - z_j)^3}$$

are corollaries of (38).

Proof. This proposition is a corollary of a more general result proved in [8]. □

It is a nontrivial fact that the Hamiltonian system of the charges interacting through two body potentials is also a corollary of a lower order dynamical system of pairwise interacting points. In general, such a property is equivalent to compatibility of a highly overdetermined system of equations. Perhaps, elliptic generalizations of (38) are the only systems satisfying such conditions [8].

It is easy to note that, when $\Lambda = 1$, equation (39) is a sum of two independent Hamiltonians

$$H = H_1 + H_2 \qquad H_1 = \sum_{i=1}^{n} \left(\frac{\mathrm{d}z_i}{\mathrm{d}t}\right)^2 - \sum_{i<j=1}^{n} \frac{2}{(z_i - z_j)^2}$$

$$H_2 = \sum_{i=n+1}^{n+m} \left(\frac{\mathrm{d}z_i}{\mathrm{d}t}\right)^2 - \sum_{i<j=n+1}^{n+m} \frac{2}{(z_i - z_j)^2}.$$

Each of them belongs to the completely integrable (in the Liouville sense) Calogero–Moser system [9]. As a corollary (38) is integrable when $\Lambda = 1$.

Equation (39) is also the Calogero–Moser Hamiltonian if $\Lambda = -1$.

In general, (39) is not reduced to known integrable Hamiltonians.

It is conjectured in [8] that (38) is integrable for any real Λ in the sense that there exist $2(n+m) - 1$ functionally independent real constants of motion which are rational functions of z_i and z_i^* (an asterisk denotes the complex conjugation and $i = 1 + \cdots + n + m$).

It can be seen from the above discussion that the case $\Lambda = 1$ has particular significance both in the problem of rational integrals (1) and from the point of view of integrable Hamiltonian systems. It is interesting to establish the role of the second case $\Lambda = 2$ in the theory of the dynamical system (38).

8. Conclusion

The principal question in this paper to be addressed is the existence and explicit representation of the Baker-Akhieser function (or equivalently $p(k, z), q(k, z)$) for $\Lambda = 2$. It is worth mentioning a useful generalization (interesting in itself) which might help to find the answer: by analogy with the $\Lambda = 1$ case, one may consider equilibrium of charges on a cylinder. Recall that (e.g. see [4, 8]) equilibrium configurations for $\Lambda = 1$ are zeros of trigonometric polynomials obtained by a finite number of Darboux transformations from the free Schrödinger operator on a circle. In such a setting, both the Adler–Moser polynomials and $\Lambda = 1$ rational

Baker–Akhieser function can be obtained by choosing different parameters in the limit when the radius of the cylinder goes to infinity.

It might be interesting to examine a similar procedure obtaining an analogue of proposition 1 for the cylinder.

Another interesting question, as mentioned before, is to understand the role of the case $\Lambda = 2$ in the theory of dynamical systems (38) and integrability of the related Hamiltonians.

Finally, we would like to mention the relation with the reduced model of superconductivity [10], the Gaudin magnet model [7] on one hand and the $\Lambda = 2$ equilibrium configurations on the other. These models are sets of commuting quantum Hamiltonians acting on a finite-dimensional Hilbert space. Equations for the Hamiltonian eigenvalues are equilibrium conditions for a set of charges of value -2 located in the complex plane, subject to mutual repulsion, and attraction of charges of value 1 located at positions defined by the parameters of the Hamiltonian. The $\Lambda = 2$ case considered in the present work corresponds to the Richardson (Gaudin) models with special restrictions imposed on the parameters of the Hamiltonian. It is interesting to examine the relations between such models and establish their special properties connected with the $\Lambda = 2$ case.

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References

[1] Adler M and Moser J 1978 On a class of polynomials connected with the Korteveg–de Vries equation *Commun. Math. Phys.* **61** 1–30

[2] Bartman A B 1984 A new interpretation of the Adler–Moser KdV polynomials: interaction of vortices *Nonlinear and Turbulent Processes in Physics* vol 3 (Chur: Harwood Academic) pp 1175–81

[3] Burchnall J L and Chaundy T W 1929 A set of differential equations which can be solved by polynomials *Proc. Lond. Soc.*

[4] Berest Y and Loutsenko I 1997 Huygens' principle in Minkowski spaces and soliton solutions of the Korteweg–de Vries equation *Commun. Math. Phys.* **190** 113–32

[5] Duistermaat J J and Grünbaum F A 1986 Differential equations in the spectral parameter *Commun. Math. Phys.* **103** 177–240

[6] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)

[7] Gaudin M 1995 États propres et valeurs propres de l'Hamiltonien d'appariement *Travaux de Michel Gaudin, Modéles exactement résolus* (France: Les Éditions de Physique)

[8] Loutsenko I 2003 Integrable dynamics of charges related to bilinear hypergeometric equation *Commun. Math. Phys.* **242** 251

[9] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220

[10] Richardson R W 1965 Exact eigenstates of the pairing-force Hamiltonian II *J. Math. Phys.* **6** 1034–51

[11] See book by Sego G and references therein *Orthogonal Polynomials* (Providence, RI: American Mathematical Society)